International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

Diagrams of Kochen–Specker Type Constructions[†]

Josef Tkadlec[1]

Received December 8, 1999

We present orthogonality diagrams of several quantum logics without any two-valued state. We present the smallest (for various notions of "smallness") known examples of such so-called Kochen–Specker-type constructions.

1. INTRODUCTION

A quantum logic is usually considered as an orthomodular lattice of closed subspaces of a Hilbert space, where closed subspaces correspond to quantum propositions. Obviously, the set of states on such a quantum logic is large (full).

An important role is played by two-valued states. [For a connection with yes–no experiments and with the hidden variable hypothesis see, e.g., Kochen and Specker (1967), Bub (1996), or Svozil and Tkadlec (1996).] According to the Gleason theorem, there is no two-valued state on the quantum logic of all closed subspaces if the dimension of the Hilbert space is at least 3. While Gleason's theorem uses substantially an infinite number of elements, Kochen and Specker (1967) presented an example of a finite quantum logic in a 3-dimensional space without a two-valued state.

We present new descriptions of several small examples of this type. While a description of a set of lines by means of points on a cube surface (Peres, 1993; Bub, 1996) gives an insight into which lines are considered, we use descriptions that give an insight into orthogonality relations (this enables an easy study of states). We use Greechie diagrams and hypergraphs and introduce so-called dual diagrams.

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

[1] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, 166 27 Prague, Czech Republic; e-mail: tkadlec@math.feld.cvut.cz.

0020-7748/00/0300-0921$18.00/0

2. EXAMPLES

Let us start with an example in a 4-dimensional space given by Peres (1991). A *hypergraph* of this quantum logic is given in Fig. 1. Points represent atoms (lines), and maximal smooth curves (line segments in special cases) represent blocks (maximal sets of mutually orthogonal lines). This quantum logic has 24 atoms (98 elements) and 22 blocks.

The situation in a 4-dimensional space is a bit complex: two atoms (e.g., 1000 and 0100) may belong to two different blocks (with 0010 and 0001, or with 0011 and $001\bar{1}$). This cannot happen in a 3-dimensional space, where we obtain a special kind of a hypergraph called a *Greechie diagram* [see, e.g., Svozil and Tkadlec (1996) for details].

The smallest known example in a 3-dimensional space was given by Bub (1996). It has 57 atoms (116 elements) and 40 blocks. An ‘almost’ Greechie diagram of this quantum logic is given on Fig. 2. ‘Almost’ means that not all atoms are marked on this diagram—only those 33 lines necessary to prove the nonexistence of a two-valued state are marked. It should be

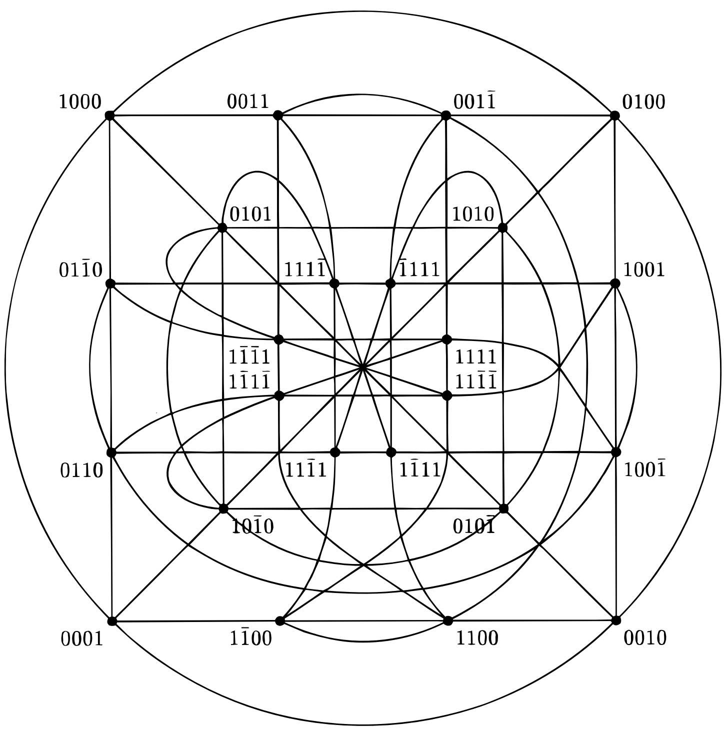

Fig. 1. A hypergraph of a quantum logic in a 4-dimensional space without a two-valued state, given by Peres (1991) [e.g., $01\bar{1}0 = \mathrm{Sp}(0, 1, -1, 0)$].

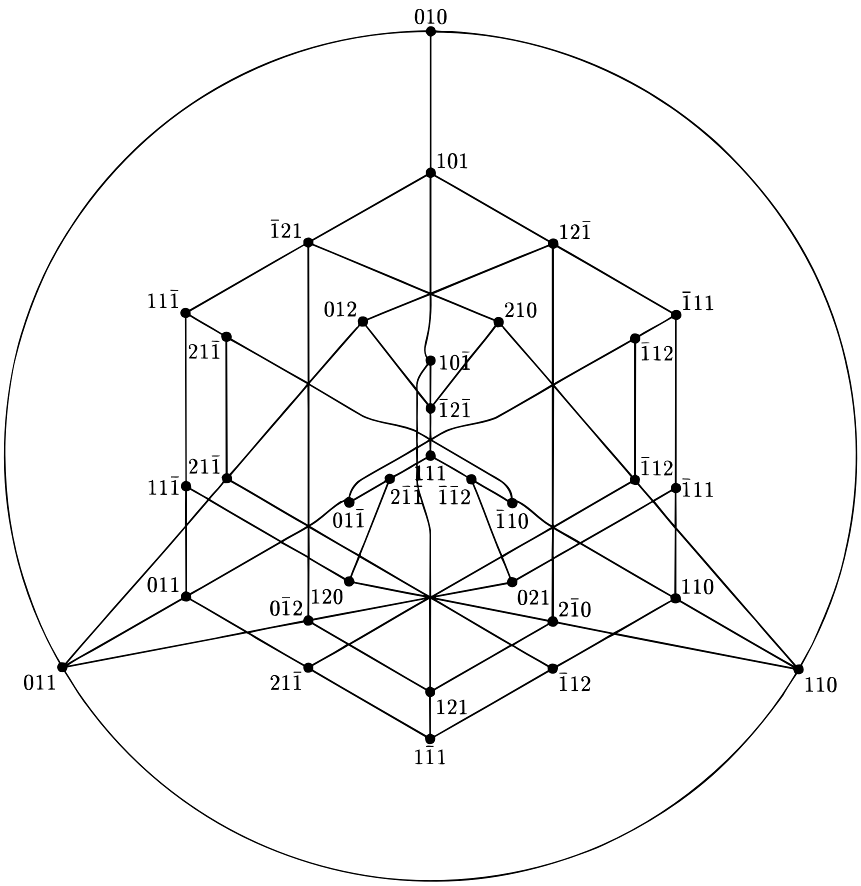

Fig. 2. 'Almost' Greechie diagram of a quantum logic in a 3-dimensional space without a two-valued state, given by Bulb (1996) [e.g., $12\bar{1}$ = Sp(1, 2, −1)].

noted that this quantum logic is a pasting of two quantum logics based on Schütte rays that do not admit a unital set of two-valued states (i.e., there is an atom such that every two-valued state takes the value 0 on it). A Greechie diagram of such a quantum logic was given by Tkadlec (1998).

Another example in a 3-dimensional space with the same number of necessary lines (but with more elements) was given by Peres (1991)—see Fig. 3 [this is a rearranged diagram from Svozil and Tkadlec (1996)]. In comparison with the example of Bub, there are more blocks. Hence it seems to be useful to use a *dual diagram*, i.e., to replace the role of points and smooth curves: points represent blocks and maximal smooth curves represent atoms. An 'almost' (not all atoms are marked) dual diagram of this example

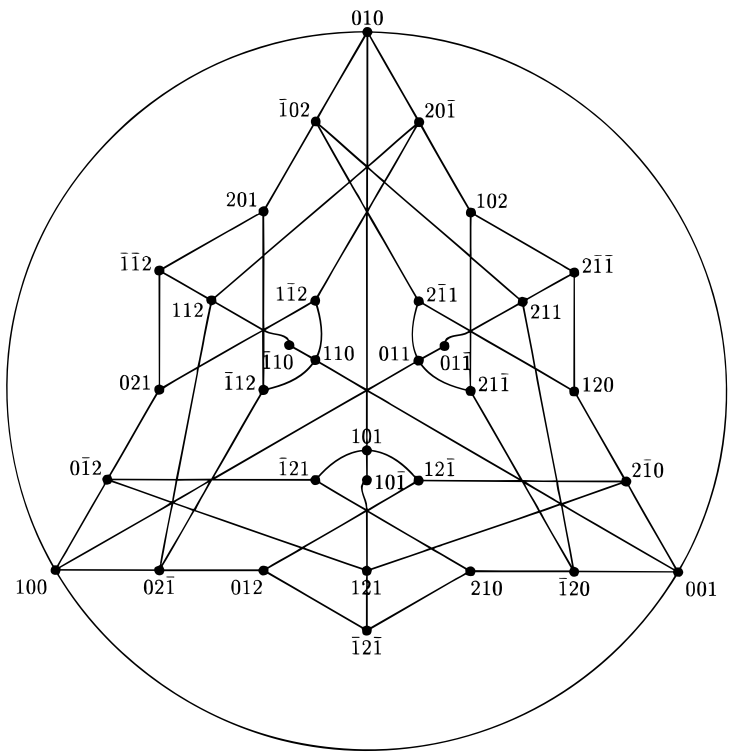

Fig. 3. 'Almost' Greechie diagram of a quantum logic in a 3-dimensional space without a two-valued state, given by Peres (1991) [e.g., $12\bar{1} = \mathrm{Sp}(1, \sqrt{2}, -1)$].

is given in Fig. 4. (If there are two curves leading from a point in some direction and two in the opposite direction, we do not consider all four maximal smooth curves, but only the two which "touch" [do not intersect] ourselves.)

The smallest known example in a 3-dimensional space in the sense of the least number of necessary lines was given by Conway and Kochen (see Peres, 1993). It has 56 atoms (112 elements) and 54 blocks, but only 31 lines and 37 blocks are necessary. An 'almost' (not all atoms and blocks are marked) dual diagram of this quantum logic is given on Fig. 5.

Table I summarizes the results for these examples.

ACKNOWLEDGMENTS

The author gratefully acknowledges the support of Grant No. 201/97/0437 of the Grant Agency of the Czech Republic and of Project No. J04/98/210000010 of the Ministry of Education of the Czech Republic.

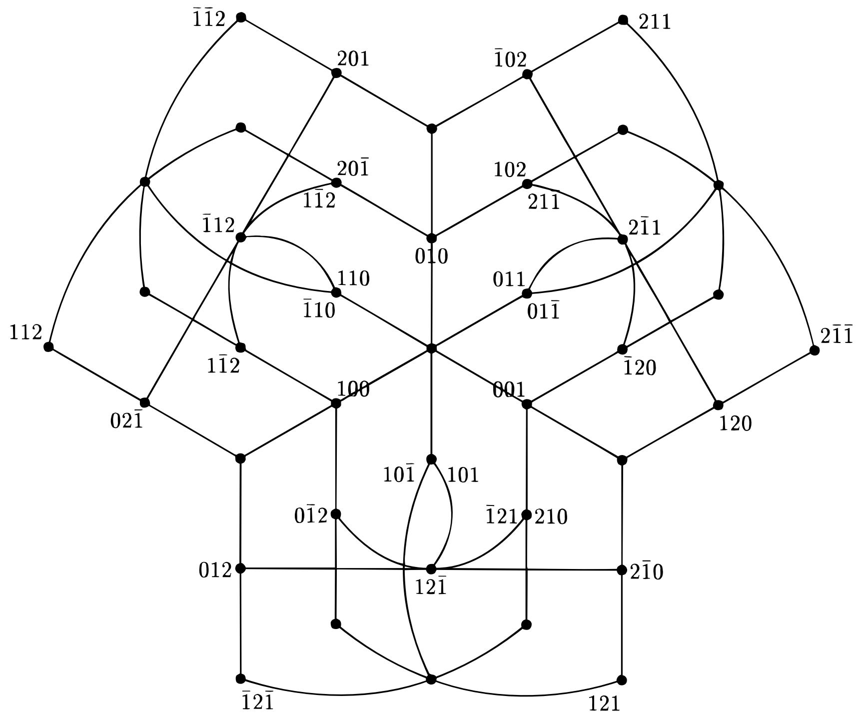

Fig. 4. 'Almost' dual diagram of a quantum logic in a 3-dimensional space without a two-valued state, given by Peres (1991) [e.g., $12\bar{1} = \mathrm{Sp}(1, \sqrt{2}, -1)$].

Table I

	$d = 4$ Peres	$d = 3$		
		Bub	Peres	Conway–Kochen
Figure	1	2	3, 4	5
Elements	98	100	116	112
Atoms	24	49	57	55
Necessary lines	24	33	33	31
Blocks	22	36	40	54
Necessary blocks	22	36	40	37

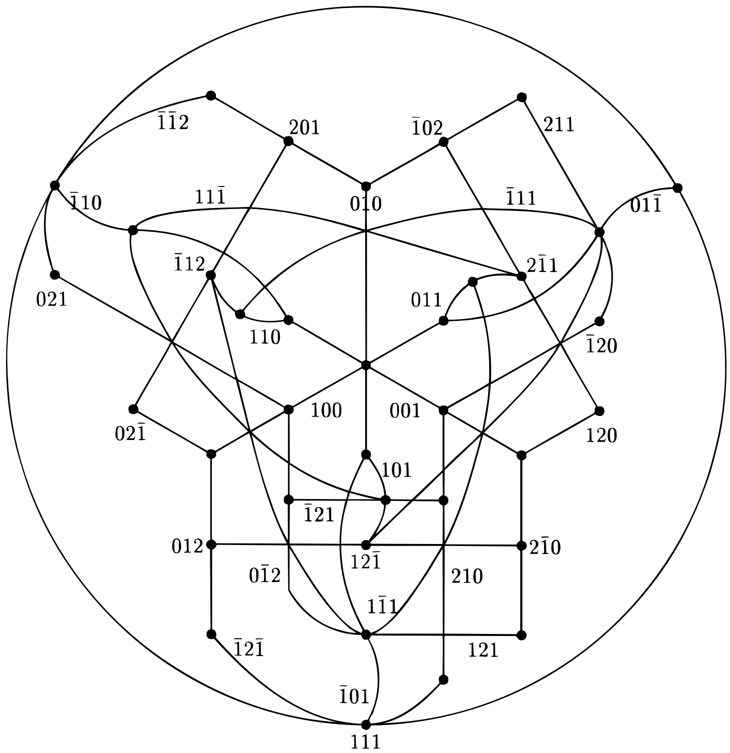

Fig. 5. 'Almost' dual diagram of a quantum logic in a 3-dimensional space without a two-valued state, given by Conway and Kochen (see Peres, 1993) [e.g., $12\bar{1} = \mathrm{Sp}(1, \sqrt{2}, -1)$].

REFERENCES

Bub, J. (1996). Schütte's tautology and the Kochen–Specker theorem, *Foundations of Physics*, **26**, 787–806.

Kochen, S., and Specker, E. P. (1967). The problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics*, **17**, 59–87.

Peres, A. (1991). Two simple proofs of the Kochen–Specker theorem, *Journal of Physics A: Mathematics and General*, **24**, L175–L178.

Peres, A. (1993). *Quantum Theory: Concepts and Methods*, Kluwer, Dordrecht.

Svozil, K., and Tkadlec, J. (1996). Greechie diagrams, nonexistence of measures and Kochen–Specker-type constructions, *Journal of Mathematical Physics*, **37**, 5380–5401.

Tkadlec, J. (1998). Greechie diagrams of small quantum logics with small state spaces, *International Journal of Theoretical Physics*, **37**, 203–209.